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Coupling, degeneracy breaking and isolation of Weibel modes in relativistic plasmas: I. General theory

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Abstract
A general proof is given that for an asymmetric particle phase-space distribution function, and in the absence of a homogeneous background magnetic field, any unstable linear Weibel modes are isolated, i.e., restricted to discrete wavenumbers. Starting from the linearized relativistic Vlasov equation it is shown that, unless the asymmetry in the distribution function is precisely zero, the broad ranges of unstable wavenumbers occurring for symmetric distribution functions are reduced to discrete, isolated wavenumbers for which unstable modes can exist. For asymmetric plasmas, electrostatic and electromagnetic wave modes are coupled to each other and the degeneracy of the two electromagnetic wave modes (that holds for symmetric distributions) is therefore broken.

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1. Introduction

The intense scrutiny of the linear Weibel [1] mode in a relativistic plasma has been a recent significant focus of research [2–8], because of the potential the mode holds to account for, or substantially influence, a plethora of astrophysical plasma situations, ranging from the creation of large-scale cosmological magnetic fields [9] via the formation of shock waves in astrophysical outflows to instabilities driven by counterstreaming electron beams in the solar system [10].

This scrutiny is because the linear Weibel mode represents a non-propagating unstable disturbance, which therefore grows in place. As has been shown recently, for asymmetric distribution functions unstable Weibel exist only for discrete wavenumbers, which we call 'isolated unstable modes'. The presence of isolated modes is very reminiscent of the development of soliton-like behaviours because the nonlinear structure of the full plasma system for a soliton also has a unique wavelike behaviour. An isolated mode in linear theory strongly suggests that the investigation of nonlinear aspects will lead to a soliton behaviour, although one must carry through the detailed quantitative development to be sure. The corresponding development of nonlinear, self-consistent Weibel-like modes is still in its infancy. Some progress has recently been reported in setting up such self-consistent modes [11], although questions of their stability to perturbations, and of their nonlinear mode-coupling to other linear or nonlinear plasma waves, still need to be addressed in detail. In principle, this approach offers a way to incorporate the effect that the unstable modes impose on the particle distribution, therefore making it also necessary to abstain from a linearized description of the system. Although this method is powerful, the mathematical details are far too complicated to allow it to be applied to general, arbitrary plasma distribution functions.

Both the linear Weibel mode and its nonlinear counterpart have been developed so far as purely transverse waves with no coupling to any electrostatic wave components. Such an evaluation of the linear mode is appropriate when the corresponding particle distribution functions are symmetric in their momentum components, and is appropriate for the nonlinear self-consistent mode when the particle distribution functions are symmetric in canonical momenta components (i.e. involving the vector potential).

However, in many situations in relativistic astrophysical plasmas one is faced with particle distributions that are manifestly asymmetric in some of their momentum components (e.g., 'jets' from active galactic nuclei (AGN)) so that there are the capabilities of (i) coupling the transverse Weibel-like modes to electrostatic mode components and (ii) also of breaking the degeneracy for the transverse linear Weibel-modes depending on the wave propagation direction relative to the particle asymmetry direction.

In this series of papers, the asymmetric linear mode situation for a homogeneous relativistic plasma in the absence of an embedded uniform magnetic field is investigated in order to evaluate both the coupling effects and the degeneracy breaking factors for Weibel-like modes that have the dependence $\exp[\mathrm{i}k(x - \mathrm{i}Mct)]$ for first-order perturbations, both k and M are taken to be real so that any mode varies in time as $\exp[kMct]$. Whereas this first part is restricted to the general theory, in the forthcoming second part specific examples for symmetric and asymmetric beam-plasma systems will be given.

Dispersion relations for relativistic plasmas have been investigated by Bret *et al* [12]. However, because they assumed a gyrotropic distribution function, the 3×3 determinant for the dispersion relation did not include coupling effects between the longitudinal and transverse modes. Only the two electromagnetic wave modes were coupled to each other. Therefore, the electrostatic and the electromagnetic wave modes were factorized and the degeneracy of the two electromagnetic modes was not broken. In this case, no isolated Weibel modes are present.

Other growth rate studies of electromagnetic instabilities at oblique angles with respect to a relativistic beam-plasma system have been carried out under the constraint of limiting the relativistic motion solely to the beam direction [13]. Additionally, in [13] the analysis was limited to a specific form of the distribution function of the beam, consisting of a waterbag distribution for the background plasma and a delta function as well as—to incorporate thermal effects—a waterbag function for the electron beam. In contrast, no constraints for the distribution functions will be assumed in this paper, so that the methods developed here are applicable to any kind of distribution function with the only restriction that the charge neutrality is satisfied (equations (1*a*) and (1*b*)).

Numerical studies of the filamentation instability [14] also confirm that the fastest growing wave mode is neither purely longitudinal nor purely transverse.

Therefore, in this paper a fully relativistic plasma is considered that is neither gyrotropic in the plane perpendicular to the wave propagation direction, nor symmetric in general. The kinetic instabilities investigated in this paper are not derivable with the methods of fluid models. Furthermore, investigations such as that of Kalman *et al* [15] are limited to the non-relativistic regime and are, therefore, not appropriate to cover the full complexity of the situation considered in this paper. The only limitation of our investigation is due to the restriction on a linear description of the investigated unstable modes. However, while it is certainly true that nonlinear effects will distort *any* linear mode, for new effects first the linear behaviour needs to be investigated, as is done in this paper.

2. The general dispersion relation

The construction of the general 3×3 determinant describing the connection between k and M is well known [16], so here the discussion is brief.

For arbitrary particle distribution functions $F_a(\boldsymbol{p})$, where a describes the particle species, and with the constraints

$$\sum_a e_a n_a \int \mathrm{d}^3 p\ F_a(\boldsymbol{p}) = 0 \tag{1a}$$

$$\sum_a e_a n_a \int \mathrm{d}^3 p\ \frac{p}{\gamma}\ F_a(\boldsymbol{p}) = 0, \tag{1b}$$

where $\gamma^2 = 1 + p^2$ denotes the squared Lorentz factor (where $\boldsymbol{p}$ is the normalized momentum defined through $\boldsymbol{p} = \boldsymbol{P}/(m_a c)$, for $\boldsymbol{P}$ the real momentum), and n_a is the equilibrium number density of each species, one can write the dispersion relation in the form [16]

$$\det \eta_{lm} = 0, \tag{2}$$

where the elements η_{lm} are given by

$$\eta_{11} = k^2 - \sum_a \xi_a^2 I_a(M) \tag{3a}$$

$$\eta_{12} = \sum_a \xi_a^2 I_{y,a}(M) \tag{3b}$$

$$\eta_{13} = \sum_a \xi_a^2 I_{z,a}(M) \tag{3c}$$

$$\eta_{21} = -\sum_a \xi_a^2 I_{y,a}(M) \tag{3d}$$

$$\eta_{22} = k^2(1 + M^2) + \sum_a \left[\xi_a^2 (g_{x,a} + g_{z,a}) + I_{yy,a}(M)\right] \tag{3e}$$

$$\eta_{23} = \sum_a \xi_a^2 [I_{yy,a}(M) - h_{yz,a}] \tag{3f}$$

$$\eta_{31} = -\sum_a \xi_a^2 I_{z,a}(M) \tag{3g}$$

$$\eta_{32} = \sum_a \xi_a^2 [I_{yz,a}(M) - h_{yz,a}] \tag{3h}$$

$$\eta_{33} = k^2(1 + M^2) + \sum_a \xi_a^2 [g_{x,a} + g_{y,a} + I_{zz,a}(M)]. \tag{3i}$$

Furthermore, definitions were introduced as

$$g_{x,a} = \int \mathrm{d}^3p\, \frac{1+p_x^2}{\gamma^3}\, F_a(\boldsymbol{p}) \tag{4a}$$

$$g_{(y,z),a} = \int \mathrm{d}^3p\, \frac{p_{y,z}^2}{\gamma^3}\, F_a(\boldsymbol{p}) \tag{4b}$$

$$h_{yz,a} = \int \mathrm{d}^3p\, \frac{p_y p_z}{\gamma^3}\, F_a(\boldsymbol{p}) \tag{4c}$$

$$I_a(M) = \int \mathrm{d}^3p\, \frac{\gamma}{p_x - \mathrm{i}M\gamma}\, \frac{\partial F_a}{\partial p_x} \tag{4d}$$

and

$$(I_{y,a};\, I_{z,a};\, I_{yz,a};\, I_{yy,a};\, I_{zz,a}) = \int \frac{\mathrm{d}^3p}{p_x - \mathrm{i}M\gamma}\, \frac{\partial F_a}{\partial p_x} \left(p_y;\, p_z;\, \frac{p_y p_z}{\gamma};\, \frac{p_y^2}{\gamma};\, \frac{p_z^2}{\gamma}\right), \tag{4e}$$

where $\xi_a^2 = 4\pi e_a^2 n_a \big/ \big(m_a c^2\big)$.

Note that the anisotropy direction of $F_a(\boldsymbol{p})$ is not necessarily parallel to the x axis—the direction of the first-order dependence $\exp[\mathrm{i}kx]$—but that the constraints on the particle anisotropy are such that equations (1*a*) and (1*b*) must be satisfied.

The analysis to follow is restricted to M real in order that one deals with temporally growing but non-propagating disturbances.

First, in figure 1, different kinds of distribution functions are sketched by means of Maxwellian distributions, that are: (a) *symmetric* in each individual momentum component p_x, p_y and p_z, and, at the same time, *gyrotropic* (i.e., symmetric around the x axis that is assumed to be the axis of wave propagation); (b) gyrotropic, but *asymmetric* in the momentum component p_x; (c) non-gyrotropic, but symmetric with respect to the momentum vector $\boldsymbol{p} = (p_x, p_y, p_z)$ and (d) non-gyrotropic and asymmetric in individual momentum components as well as in the momentum vector.

3. Symmetric situations

3.1. Individual momentum components

When each $F_a(\boldsymbol{p})$ is symmetric *separately* in each of p_x, p_y, p_z then

$$h_{yz,a} = I_{y,a} = I_{z,a} = I_{yz,a} = 0. \tag{5}$$

In this case only the diagonal terms of the 3×3 determinant survive so that one has three independent modes described, respectively, by

$$k^2 = \sum_a \xi_a^2 I_a(M) \tag{6a}$$

$$k^2 = -\sum_a \xi_a^2 [g_{x,a} + g_{y,a} + I_{yy,a}(M)](1+M^2)^{-1} \tag{6b}$$

$$k^2 = -\sum_a \xi_a^2 [g_{x,a} + g_{y,a} + I_{zz,a}(M)](1+M^2)^{-1}. \tag{6c}$$

One recognizes the mode described by equation (6*a*) as the conventional electrostatic mode, while the modes described by equations (6*b*) and (6*c*) correspond to the electromagnetic

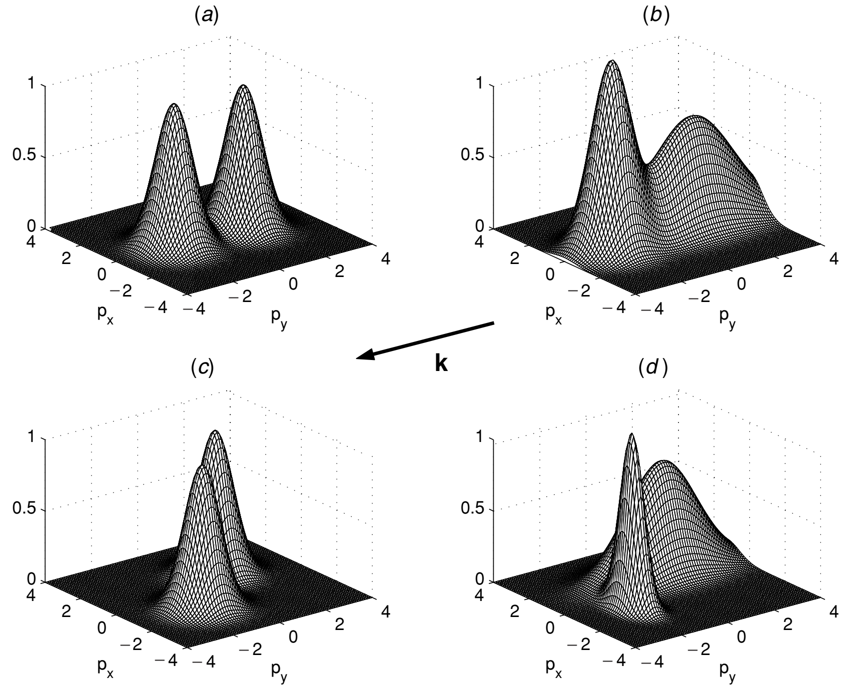

Figure 1. Different types of symmetric distribution functions. For simplicity, the p_z direction is not shown and assumed to have the form $\exp[-p_z^2]$. Therefore, the distribution function in figure (*a*) is rotational invariant around the x axis, which is the axis of wave propagation (indicated by the arrow denoting the wave vector $\boldsymbol{k}$); this is called 'gyrotropic'. Furthermore, this function is symmetric in all three momentum components. The distribution in figure (*b*) is also gyrotropic, but asymmetric in p_y. The distribution in figure (*c*) is neither gyrotropic nor symmetric in p_x or p_y. But it has total momentum symmetry, i.e., it is symmetric if $\boldsymbol{p} \to -\boldsymbol{p}$. The distribution in figure (*d*), finally, is non-gyrotropic and asymmetric in individual momentum components as well as in the total momentum.

Weibel modes with degeneracy if and only if the distribution functions are cylindrically symmetric in the momentum components perpendicular to the x axis, i.e., $F_a(\boldsymbol{p}) = F_a\big(p_x^2, p_y^2 + p_z^2\big)$. For other than cylindrical symmetry there are two separate electromagnetic modes.

3.2. Total momentum symmetry

When each $F_a(\boldsymbol{p})$ is symmetric in the sense that $F_a(-\boldsymbol{p}) = F_a(\boldsymbol{p})$, a weaker symmetry than for the individual component situation described above, then the factors $I_{y,a}$ and $I_{z,y}$ coupling the electromagnetic component of the 3×3 determinant are not zero in general, and the off-diagonal electromagnetic components are also not zero. In this situation there is not only degeneracy lifting of the electromagnetic components of the determinant but also a coupling to the electrostatic component. Conversely, there is also a coupling of the diagonal electrostatic component to the electromagnetic components.

Note that there are now the pure imaginary components

$$
\begin{aligned}
(I_{y,a}; I_{z,a}) &= \mathrm{i}M \int \mathrm{d}^3 p \, \frac{\gamma}{p_x^2 + M^2\gamma^2} \, \frac{\partial F_a}{\partial p_x}(p_y; p_z) \\
&\equiv \mathrm{i}M(J_{x,a}; J_{y,a})
\end{aligned}
\tag{7a}
$$

together with the real components

$$(I_{yz,a};\, I_{yy,a};\, I_{zz,a}) = \int \mathrm{d}^3p\, \frac{p_x}{\gamma\left(p_x^2 + M^2\gamma^2\right)} \frac{\partial F_a}{\partial p_x}\left(p_y p_z;\, p_y^2;\, p_z^2\right) \tag{7b}$$

and

$$I_a(M) = \int \mathrm{d}^3p\, \frac{\gamma p_x}{p_x^2 + M^2\gamma^2} \frac{\partial F_a}{\partial p_x}. \tag{7c}$$

However, from the structure of the 3×3 determinant one notes that the pure imaginary components always occur in product forms of the sort $\left(\sum_a I_{y,a}\right)\cdot\left(\sum_a I_{y,a}\right)$, $\left(\sum_a I_{y,a}\right)\cdot\left(\sum_a I_{z,a}\right)$ or $\left(\sum_a I_{z,a}\right)\cdot\left(\sum_a I_{z,a}\right)$ so that they provide *real* contributions to the 3×3 determinant.

Accordingly, write the real factors

$$E = \sum_a \xi_a^2 I_a(M) \tag{8a}$$

$$Y = \sum_a \xi_a^2[g_{x,a} + g_{z,a} + I_{zz,a}(M)] \tag{8b}$$

$$Z = \sum_a \xi_a^2[g_{x,a} + g_{y,a} + I_{yy,a}(M)] \tag{8c}$$

$$D = -\sum_a \xi_a^2[h_{yz,a} - I_{yz,a}(M)] \tag{8d}$$

and the pure imaginary factors

$$\mathrm{i}C_y = \sum_a \xi_a^2 I_{y,a}(M) = \mathrm{i}M\sum_a \xi_a^2 J_{y,a}(M) \tag{8e}$$

$$\mathrm{i}C_z = \sum_a \xi_a^2 I_{z,a}(M) = \mathrm{i}M\sum_a \xi_a^2 J_{z,a}(M) \tag{8f}$$

so that the dispersion relation (2) takes the form

$$\begin{vmatrix} k^2 - E & \mathrm{i}C_y & \mathrm{i}C_z \\ -\mathrm{i}C_y & k^2(1+M^2)+Y & D \\ -\mathrm{i}C_z & D & k^2(1+M^2)+Z \end{vmatrix} = 0. \tag{9}$$

Then one has

$$\begin{aligned} 0 = {} & (k^2 - E)[k^2(1+M^2)+Y][k^2(1+M^2)+Z] - D^2(k^2 - E^2) \\ & - C_y^2[k^2(1+M^2)+Z] - C_z^2[k^2(1+M^2)+Y] + 2DC_yC_z \end{aligned} \tag{10}$$

providing a cubic equation in k^2 as a function of M. Roots of the cubic equation yielding positive values of k^2 represent the range (or ranges) of real M values permitting Weibel-type modes.

4. Asymmetric plasmas and isolated Weibel-like modes

This subsection provides a general proof that for *any* asymmetric plasma distributions the mode solutions of the 3×3 determinant from equation (2) provide solely isolated values of the phase velocity M and, associated with these isolated phase velocities, only isolated and discrete wavenumbers. There is *no* continuum range of k values, provided only that the plasma asymmetry is not *precisely* zero, i.e., finite no matter how small. Only when the asymmetry is precisely zero does one recover the continuum behaviours sketched above for symmetric plasma distribution functions.

For arbitrary plasma distribution functions, $F_a(\boldsymbol{p})$, split them into their symmetric and anti-symmetric parts with

$$F_a^{S,A}(\boldsymbol{p}) = \tfrac{1}{2}\left[F_a(\boldsymbol{p}) \pm F_a(-\boldsymbol{p})\right], \tag{11}$$

where the '+' sign denotes F_a^S and the '−' sign denotes F_a^A. Then, in the 3×3 determinant of equation (2) one can write

$$\begin{vmatrix} k^2 - \Lambda & C_y & C_z \\ -C_y & k^2 + Y & D \\ -C_z & D & k^2 + Z \end{vmatrix} = 0, \tag{12}$$

where now one has

$$\Lambda = \Lambda^R + \mathrm{i}\Lambda^I \equiv \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{\gamma p_x}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^S}{\partial p_x} + \mathrm{i}M \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{\gamma^2}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^A}{\partial p_x} \tag{13a}$$

$$C_{y,z} = C_{y,z}^R + \mathrm{i}C_{y,z}^I \equiv \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{p_x p_{y,z}}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^A}{\partial p_x} + \mathrm{i}M \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{p_{y,z}}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^S}{\partial p_x} \tag{13b}$$

$$\begin{aligned} D = D^R + \mathrm{i}D^I \equiv& \sum_a \xi_a^2 \left[-h_{yz,a} + \int \mathrm{d}^3 p \, \frac{p_x p_y p_z}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^S}{\partial p_x} \right] \\ &+ \mathrm{i}M \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{p_y p_z}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^A}{\partial p_x} \end{aligned} \tag{13c}$$

$$\begin{aligned} Y = Y^R + \mathrm{i}Y^I \equiv& \sum_a \xi_a^2 \left[g_{x,a} + g_{z,a} + \int \mathrm{d}^3 p \, \frac{p_x p_y^2}{\gamma\left(p_x^2 + M^2\gamma^2\right)} \frac{\partial F_a^S}{\partial p_x} \right] \\ &+ \mathrm{i}M \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{p_y^2}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^A}{\partial p_x} \end{aligned} \tag{13d}$$

$$\begin{aligned} Z = Z^R + \mathrm{i}Z^I \equiv& \sum_a \xi_a^2 \left[g_{x,a} + g_{y,a} + \int \mathrm{d}^3 p \, \frac{p_x p_z^2}{\gamma\left(p_x^2 + M^2\gamma^2\right)} \frac{\partial F_a^S}{\partial p_x} \right] \\ &+ \mathrm{i}M \sum_a \xi_a^2 \int \mathrm{d}^3 p \, \frac{p_z^2}{p_x^2 + M^2\gamma^2} \frac{\partial F_a^A}{\partial p_x} \end{aligned} \tag{13e}$$

with superscripts R and I representing real and imaginary components, respectively.

One can then evaluate the 3×3 determinant from equation (12) and set the real and imaginary components of the determinant to zero separately under the proviso that k^2 and M are considered real. Then the real part of the dispersion relation, equation (12), yields an equation cubic in k^2, as

$$\begin{aligned} 0 =& (k^2 - \Lambda^R)[(k^2 + Y^R)(k^2 + Z^R) - Y^I Z^I] + \Lambda^I[Y^I(k^2 + Z^R) + Z^I(k^2 + Y^R)] \\ &- 2D^R\left(C_y^R C_z^R - C_y^I C_z^I\right) + 2D^I\left(C_y^I C_z^R + C_y^R C_z^I\right) \\ &+ (k^2 + Z^R)\left(C_y^R C_y^R - C_y^I C_y^I\right) - 2Z^I C_y^R C_y^I + (k^2 + Y^R)\left(C_z^R C_z^R - C_z^I C_z^I\right) \\ &- 2Y^I C_z^R C_z^I - (k^2 - \Lambda^R)(D^R D^R - D^I D^I) - 2\Lambda^R D^R D^I. \end{aligned} \tag{14a}$$

For later use, write equation (14*a*) in the generic form

$$0 = a(M)k^6 + b(M)k^4 + d(M)k^2 + f(M). \tag{14b}$$

The imaginary part of the 3×3 determinant yields an equation quadratic in k^2 as

$$\begin{aligned}0 = &-\Lambda^I[(k^2+Y^R)(k^2+Z^R) - Y^I Z^I] + (k^2-\Lambda^R)[Y^I(k^2+Z^R) + Z^I(k^2+Y^R)]\\ &-2D^R\left[C_y^I C_z^R + C_y^R C_z^I\right] - 2D^I\left(C_y^R C_z^R - C_y^I C_z^I\right) + Z^I\left(C_y^R C_y^R - C_y^I C_y^I\right)\\ &+2C_y^R C_y^I(k^2+Z^R) + Y^I\left(C_z^R C_z^R - C_z^I C_z^I\right) + 2C_z^R C_z^I(k^2+Y^R)\\ &-2D^R D^I(k^2-\Lambda^R) + 2\Lambda^I(D^R D^R - D^I D^I).\end{aligned} \tag{15a}$$

For later use, write equation (15*a*) in the generic form

$$0 = \tilde{b}(M)k^4 + \tilde{d}(M)k^2 + \tilde{f}(M). \tag{15b}$$

But both equations (14*b*) and (15*b*) must be satisfied. So eliminate k^2 between the pair of equations (14*b*) and (15*b*). This elimination is most simply achieved by using equation (15*b*) in the form $k^4 = -[\tilde{f}(M)/\tilde{b}(M) + \tilde{d}(M)k^2/\tilde{b}(M)]$ repeatedly in equation (14*b*) to reduce equation (14*b*) to the form

$$k^2 = \frac{\eta(M)}{\zeta(M)}, \tag{16}$$

where

$$\eta(M) = \frac{\tilde{f}(M)b(M)}{\tilde{b}(M)} - \frac{\tilde{d}(M)\tilde{f}(M)a(M)}{\tilde{b}(M)^2} - f(M) \tag{17a}$$

$$\zeta(M) = \frac{\tilde{d}(M)^2 a(M)}{\tilde{b}(M)^2} - \frac{\tilde{f}(M)a(M)}{\tilde{b}(M)} - \frac{\tilde{d}(M)b(M)}{\tilde{b}(M)} + d(M). \tag{17b}$$

Then insert equation (16) into equation (15*b*) to eliminate k^2 and so finally obtain

$$\tilde{b}(M)\eta(M)^2 + \tilde{d}(M)\eta(M)\zeta(M) + \tilde{f}(M)\zeta(M)^2 = 0. \tag{18}$$

Hence the theorem is proven because equation (18) provides discrete values of M (that *must* be real and that are determined solely by the plasma parameters) and which, when each is used in equation (16), provide *fixed* values of k^2 (that must also be real and positive).

Hence for *any* asymmetric plasma distribution functions (subject only to the constraints of equations (1*a*) and (1*b*) in order to have a charge and current neutral original plasma) any modes of the form $\exp[ikx + kMct]$ with k and M both real are isolated, discrete modes.

The mode wave phase speed M is to be determined from the (finite) real M solutions of equation (18) and the corresponding discrete k^2 values from the solutions given through equation (16) that yield $k^2 > 0$, i.e., $\eta(M)\zeta(M) > 0$. These results are valid for any asymmetry, no matter how small (unless the asymmetry is precisely zero). Hence, all Weibel-like modes of an asymmetric plasma are isolated modes in the absence of an embedded background magnetic field.

5. Discussion and conclusion

In this paper, linear purely growing instabilities in a relativistic plasma have been investigated in the absence of a homogeneous background magnetic field. Because the investigation was undertaken in the absence of a homogeneous magnetic field in order to evaluate both the coupling effects and the degeneracy breaking factors for Weibel-like modes, the propagation direction of the unstable waves was, without loss of generality, limited to only one direction.

It is well known that for a symmetric particle distribution function broad wavenumber ranges can be found that permit unstable wave modes. In contrast, for an asymmetric particle distribution function it has been now generally proven here that any unstable Weibel modes are isolated. This result holds unless the asymmetry is precisely zero, showing that the broad range of unstable wavenumbers occurring for symmetric distribution functions collapses into discrete wavenumbers that permit unstable modes.

Presumably, the countable set of discrete modes is somewhat similar in character to the countable set of discrete modes one obtains with a multi-beam plasma. As the number of beams becomes large without limit the count of discrete modes in each momentum interval also becomes large but so that a fixed fraction of the total countable set is contained in each momentum interval. Thus one obtains the continuum representation of the modes. It can be speculated that the same is true for the discrete and isolated Weibel-type modes as the anisotropy tends to zero, although this remains to be investigated in detail.

Furthermore, for asymmetric plasmas, electrostatic and electromagnetic wave modes are coupled to each other. In this case, the degeneracy of the two electromagnetic wave modes is broken.

Because, in previous works [12–14] specific distribution functions have been assumed that were either gyrotropic or symmetric in the direction of the wave propagation, it was not possible to show the occurrence of isolated Weibel modes. In this paper, in contrast, the whole complexity of a relativistic three-dimensional plasma with no requirements on the distribution function has been investigated. Therefore, independent of the form of the distribution function, whenever there is an asymmetry in the distribution function, any unstable linear Weibel modes are restricted to discrete wavenumbers.

The question if such isolated modes really exist, however, has not been addressed in this first paper. Only if the equation that determines the product of growth rate and wavenumber, equation (18), has real and positive solutions, it has been shown that the resulting unstable modes *must* be isolated. In a second paper, illustrative examples and numerical estimates of the growth rates will be given for asymmetric distribution functions that, as will be shown show, indeed allow the existence of such isolated unstable modes. Furthermore, self-consistent particle-in-cell (PIC) simulations are currently in progress to confirm the existence of isolated Weibel modes in asymmetric particle distribution functions.

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